In this chapter, you will explore the different relationships formed by parallel and perpendicular lines and planes. Different angle relationships will also be explored and what happens to these angles when lines are parallel. You will continue to use proofs, to prove that lines are parallel or perpendicular. There will also be a review of equations of lines and slopes and how we show algebraically that lines are parallel and perpendicular.
3.1 Lines and Angles

Learning Objectives

- Identify parallel lines, skew lines, and parallel planes.
- Use the Parallel Line Postulate and the Perpendicular Line Postulate.
- Identify angles made by transversals.

Review Queue

1. What is the equation of a line with slope -2 and passes through the point (0, 3)?
2. What is the equation of the line that passes through (3, 2) and (5, -6).
3. Change $4x - 3y = 12$ into slope-intercept form.
4. Are $y = \frac{1}{3}x$ and $y = -3x$ perpendicular? How do you know?

Know What? A partial map of Washington DC is shown. The streets are designed on a grid system, where lettered streets, A through Z run east to west and numbered streets 1st to 30th run north to south. Just to mix things up a little, every state has its own street that runs diagonally through the city. There are, of course other street names, but we will focus on these three groups for this chapter. Can you explain which streets are parallel and perpendicular? Are there any that seem to be neither? How do you know these streets are parallel or perpendicular?

If you are having trouble viewing this map, check out the interactive map here: http://www.travelguide.tv/washington/map.html
3.1 Lines and Angles

Defining Parallel and Skew

**Parallel:** Two or more lines that lie in the same plane and never intersect.

The symbol for parallel is $\parallel$. To mark lines parallel, draw arrows ($\geq$) on each parallel line. If there are more than one pair of parallel lines, use two arrows ($\gg$) for the second pair. The two lines to the right would be labeled $\overrightarrow{AB} \parallel \overrightarrow{MN}$ or $\overline{l} \parallel \overline{m}$.

![Parallel Lines Diagram]

Planes can also be parallel or perpendicular. The image to the left shows two parallel planes, with a third blue plane that is perpendicular to both of them.

![Parallel Planes Diagram]

An example of parallel planes could be those containing the top of a table and the floor. The legs would be in perpendicular planes to the table top and the floor.

**Skew lines:** Lines that are in different planes and never intersect.

**Example 1:** In the cube, list:

![Cube Diagram]

a) 3 pairs of parallel planes
b) 2 pairs of perpendicular planes

c) 3 pairs of skew line segments

**Solution:**

a) Planes $ABC$ and $EFG$, Planes $AEG$ and $FBH$, Planes $AEB$ and $CDH$

b) Planes $ABC$ and $CDH$, Planes $AEB$ and $FBH$ (there are others, too)

c) $BD$ and $CG$, $BF$ and $EG$, $GH$ and $AE$ (there are others, too)

**Parallel Line Postulate**

**Parallel Line Postulate:** For a line and a point not on the line (we call this an *external* point), there is exactly one line parallel to this line through the point.

There are infinitely many lines that pass through $A$, but only one is parallel to $l$.

**Investigation 3-1: Patty Paper and Parallel Lines**

1. Get a piece of patty paper (a translucent square piece of paper).

   Draw a line and a point above the line.

2. Fold up the paper so that the line is over the point. Crease the paper and unfold.

3. Are the lines parallel? Yes, by design, this investigation replicates the line we drew in #1 over the point. Therefore, there is only one line parallel through this point to this line.
Perpendicular Line Postulate

**Perpendicular Line Postulate:** For a line and a point not on the line, there is exactly one line perpendicular to the line that passes through the point.

There are infinitely many lines that pass through $A$, but only one that is perpendicular to $l$.

**Investigation 3-2: Perpendicular Line Construction; through a Point NOT on the Line (an *external point*)**

1. Draw a horizontal line and a point above that line.

Label the line $l$ and the point $A$.

![Diagram of line $l$ and point $A$.]

2. Take the compass and put the pointer on $A$. Open the compass so that it reaches beyond line $l$. Draw an arc that intersects the line twice.

![Diagram showing two arcs intersecting the line.]

3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc below the line. Repeat this on the other side so that the two arc marks intersect.

![Diagram showing two arcs intersecting below the line.]

4. Take your straightedge and draw a line from point $A$ to the arc intersections below the line. This line is perpendicular to $l$ and passes through $A$. 

![Diagram showing the perpendicular line drawn through $A$.]
Notice that this is a different construction from a perpendicular bisector.

To see a demonstration of this construction, go to: http://www.mathsisfun.com/geometry/construct-perpnotline.htm

Investigation 3-3: Perpendicular Line Construction; through a Point on the Line (an internal point)

1. Draw a horizontal line and a point on that line.
   Label the line $l$ and the point $A$.

2. Take the compass and put the pointer on $A$. Open the compass so that it reaches out horizontally along the line. Draw two arcs that intersect the line on either side of the point.

3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc above or below the line. Repeat this on the other side so that the two arc marks intersect.

4. Take your straightedge and draw a line from point $A$ to the arc intersections above the line. This line is perpendicular to $l$ and passes through $A$. 
Notice that this is a different construction from a perpendicular bisector.

To see a demonstration of this construction, go to: http://www.mathsisfun.com/geometry/construct-perponline.html

Angles and Transversals

Transversal: A line that intersects two distinct lines. These two lines may or may not be parallel.

The area between \( l \) and \( m \) is the called the interior. The area outside \( l \) and \( m \) is called the exterior.

Looking at \( t, l \) and \( m \), there are 8 angles formed and several linear pairs vertical angle pairs. There are also 4 new angle relationships, defined here:

Corresponding Angles: Two angles that are in the same place with respect to the transversal, but on different lines. Imagine sliding the four angles formed with line \( l \) down to line \( m \). The angles which match up are corresponding. \( \angle 2 \) and \( \angle 6 \) are corresponding angles.

Alternate Interior Angles: Two angles that are on the interior of \( l \) and \( m \), but on opposite sides of the transversal. \( \angle 3 \) and \( \angle 6 \) are alternate interior angles.

Alternate Exterior Angles: Two angles that are on the exterior of \( l \) and \( m \), but on opposite sides of the transversal. \( \angle 1 \) and \( \angle 8 \) are alternate exterior angles.
**Same Side Interior Angles:** Two angles that are on the same side of the transversal and on the interior of the two lines. \( \angle 3 \) and \( \angle 5 \) are same side interior angles. These are also called **consecutive interior angles**.

**Example 2:** Using the picture above, list all the other pairs of each of the newly defined angle relationships.

**Solution:**

- Corresponding Angles: \( \angle 3 \) and \( \angle 7 \), \( \angle 1 \) and \( \angle 5 \), \( \angle 4 \) and \( \angle 8 \)
- Alternate Interior Angles: \( \angle 4 \) and \( \angle 5 \)
- Alternate Exterior Angles: \( \angle 2 \) and \( \angle 7 \)
- Same Side Interior Angles: \( \angle 4 \) and \( \angle 6 \)

**Example 3:** If \( \angle 2 = 48^\circ \) (in the picture above), what other angles do you know?

**Solution:** \( \angle 2 \cong \angle 3 \) by the Vertical Angles Theorem, so \( m\angle 3 = 48^\circ \). \( \angle 2 \) is also a linear pair with \( \angle 1 \) and \( \angle 4 \), so it is supplementary to those two. They are both \( 132^\circ \). We do not know the measures of \( \angle 5 \), \( \angle 6 \), \( \angle 7 \), or \( \angle 8 \) because we do not have enough information.

**Example 4:** For the picture shown, determine:

a) A corresponding angle to \( \angle 3 \)?

b) An alternate interior angle to \( \angle 7 \)?

c) An alternate exterior angle to \( \angle 4 \)?

**Solution:** The corresponding angle to \( \angle 3 \) is \( \angle 1 \). The alternate interior angle to \( \angle 7 \) is \( \angle 2 \). And, the alternate exterior angle to \( \angle 4 \) is \( \angle 5 \).

**Know What? Revisited** For Washington DC, all of the lettered streets are parallel, as are all of the numbered streets. The lettered streets are perpendicular to the numbered streets. There are no skew streets because all of the streets are in the same plane. We also do not know if any of the state-named streets are parallel or perpendicular.

**Review Questions**

Use the figure below to answer questions 1-5. The planes containing the two pentagons are parallel and all of the rectangular sides are in planes that are perpendicular to both of the parallel planes.
1. Find two pairs of skew lines.
2. List a pair of parallel lines.
3. List a pair of perpendicular lines.
4. For $AB$, how many perpendicular lines pass through point $V$? What line is this?
5. For $XY$, how many parallel lines pass through point $D$? What line is this?

For questions 6-12, use the picture below.

6. What is the corresponding angle to $\angle 4$?
7. What is the alternate interior angle with $\angle 5$?
8. What is the corresponding angle to $\angle 8$?
9. What is the alternate exterior angle with $\angle 7$?
10. What is the alternate interior angle with $\angle 4$?
11. What is the same side interior angle with $\angle 3$?
12. What is the corresponding angle to $\angle 1$?

Use the picture below for questions 13-16.

13. If $m\angle 2 = 55^\circ$, what other angles do you know?
14. If $m\angle 5 = 123^\circ$, what other angles do you know?
15. If $t \perp l$, is $t \perp m$? Why or why not?
16. Is $l \parallel m$? Why or why not?
17. **Construction** Draw a line and a point not on the line. Construct a perpendicular line to your original line through your point.
18. **Construction** Construct a perpendicular line to the line you constructed in #12. Use the point you originally drew, so that you will be constructing a perpendicular line through a point on the line.
19. Can you use patty paper to do the construction in number 17? Draw a line and a point not on the line on a piece of patty paper (or any thin white paper or tracing paper). Think about how you could make a crease in the paper that would be a line perpendicular to your original line through your point.
20. Using what you discovered in number 19, use patty paper to construct a line perpendicular to a given line through a point on the given line.
21. Draw a pair of parallel lines using your ruler. Describe how you did this.
22. Draw a pair of perpendicular lines using your ruler. Describe your method.

Geometry is often apparent in nature. Think of examples of each of the following in nature.

23. Parallel Lines or Planes
24. Perpendicular Lines or Planes
25. Skew Lines

**Algebra Connection** In questions 26-35 we will begin to explore the concepts of parallel and perpendicular lines in the coordinate plane.

26. Write the equations of two lines parallel to $y = 3$.
27. Write the equations of two lines perpendicular to $y = 5$.
28. What is the relationship between the two lines you found for number 27?
29. Plot the points $A(2, -5)$, $B(-3, 1)$, $C(0, 4)$, $D(-5, 10)$. Draw the lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$. What are the slopes of these lines? What is the geometric relationship between these lines?
30. Plot the points $A(2, 1)$, $B(7, -2)$, $C(2, -2)$, $D(5, 3)$. Draw the lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$. What are the slopes of these lines? What is the geometric relationship between these lines?
31. Based on what you discovered in numbers 29 and 30, can you make a conjecture about the slopes of parallel and perpendicular lines?

Find the equation of the line that is **parallel** to the given line and passes through (5, -1).

32. $y = 2x - 7$
33. $y = -\frac{3}{5}x + 1$

Find the equation of the line that is **perpendicular** to the given line and passes through (2, 3).

34. $y = \frac{2}{3}x - 5$
35. $y = -\frac{3}{4}x + 9$

**Review Queue Answers**

1. $y = -2x + 3$
2. $y = -4x + 14$
3. $y = \frac{4}{3}x - 4$
4. Yes, the lines are perpendicular. The slopes are reciprocals and have opposite signs.
3.2 Properties of Parallel Lines

Learning Objectives

- Use the Corresponding Angles Postulate.
- Use the Alternate Interior Angles Theorem.
- Use the Alternate Exterior Angles Theorem.
- Use Same Side Interior Angles Theorem.

Review Queue

Use the picture below to determine:

1. A pair of corresponding angles.
2. A pair of alternate interior angles.
3. A pair of same side interior angles.
4. If $m\angle 4 = 37^\circ$, what other angles do you know?

Know What? The streets below are in Washington DC. The red street is R St. and the blue street is Q St. These two streets are parallel. The transversals are: Rhode Island Ave. (green) and Florida Ave. (orange).

1. If $m\angle FTS = 35^\circ$, determine the other angles that are $35^\circ$.
2. If $m\angle SQV = 160^\circ$, determine the other angles that are $160^\circ$. 
3. Why do you think the "state" streets exist? Why aren’t all the streets parallel or perpendicular?

In this section, we are going to discuss a specific case of two lines cut by a transversal. The two lines are now going to be parallel. If the two lines are parallel, all of the angles, corresponding, alternate interior, alternate exterior and same side interior have new properties. We will begin with corresponding angles.

**Corresponding Angles Postulate**

**Corresponding Angles Postulate:** If two parallel lines are cut by a transversal, then the corresponding angles are congruent.

If \( l \parallel m \) and both are cut by \( t \), then \( \angle 1 \cong \angle 5 \), \( \angle 2 \cong \angle 6 \), \( \angle 3 \cong \angle 7 \), and \( \angle 4 \cong \angle 8 \).

\( l \) must be parallel to \( m \) in order to use this postulate. Recall that a postulate is very much like a theorem, but does not need to be proven. We can take it as true and use it just like a theorem from this point.

**Investigation 3-4: Corresponding Angles Exploration**

You will need: paper, ruler, protractor

1. Place your ruler on the paper. On either side of the ruler, draw lines, 3 inches long. This is the easiest way to ensure that the lines are parallel.

2. Remove the ruler and draw a transversal. Label the eight angles as shown.
3. Using your protractor, measure all of the angles. What do you notice?

In this investigation, you should see that $m\angle 1 = m\angle 4 = m\angle 5 = m\angle 8$ and $m\angle 2 = m\angle 3 = m\angle 6 = m\angle 7$. $\angle 1 \cong \angle 4$, $\angle 5 \cong \angle 8$ by the Vertical Angles Theorem. By the Corresponding Angles Postulate, we can say $\angle 1 \cong \angle 5$ and therefore $\angle 1 \cong \angle 8$ by the Transitive Property. You can use this reasoning for the other set of congruent angles as well.

**Example 1:** If $m\angle 2 = 76^\circ$, what is $m\angle 6$?

**Solution:** $\angle 2$ and $\angle 6$ are corresponding angles and $l \parallel m$, from the markings in the picture. By the Corresponding Angles Postulate the two angles are equal, so $m\angle 6 = 76^\circ$.

**Example 2:** Using the measures of $\angle 2$ and $\angle 6$ from Example 2, find all the other angle measures.

**Solution:** If $m\angle 2 = 76^\circ$, then $m\angle 1 = 180^\circ - 76^\circ = 104^\circ$ because they are a linear pair. $\angle 3$ is a vertical angle with $\angle 2$, so $m\angle 3 = 76^\circ$. $\angle 1$ and $\angle 4$ are vertical angles, so $m\angle 4 = 104^\circ$. By the Corresponding Angles Postulate, we know $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$, so $m\angle 5 = 104^\circ$, $m\angle 6 = 76^\circ$, $m\angle 7 = 76^\circ$, and $m\angle 104^\circ$.

**Alternate Interior Angles Theorem**

**Example 3:** Find $m\angle 1$.

**Solution:** $m\angle 2 = 115^\circ$ because they are corresponding angles and the lines are parallel. $\angle 1$ and $\angle 2$ are vertical angles, so $m\angle 1 = 115^\circ$ also.

$\angle 1$ and the $115^\circ$ angle are alternate interior angles.

**Alternate Interior Angles Theorem:** If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.
Proof of Alternate Interior Angles Theorem

Given: \( l \parallel m \)

Prove: \( \angle 3 \cong \angle 6 \)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( l \parallel m )</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( \angle 3 \cong \angle 7 )</td>
<td>Corresponding Angles Postulate</td>
</tr>
<tr>
<td>3. ( \angle 7 \cong \angle 6 )</td>
<td>Vertical Angles Theorem</td>
</tr>
<tr>
<td>4. ( \angle 3 \cong \angle 6 )</td>
<td>Transitive PoC</td>
</tr>
</tbody>
</table>

There are several ways we could have done this proof. For example, Step 2 could have been \( \angle 2 \cong \angle 6 \) for the same reason, followed by \( \angle 2 \cong \angle 3 \). We could have also proved that \( \angle 4 \cong \angle 5 \).

**Example 4:** *Algebra Connection* Find the measure of the angle and \( x \).

**Solution:** The two given angles are alternate interior angles, so their measures are equal. Set the two expressions equal to each other and solve for \( x \).

\[
(4x - 10)° = 58°
\]
\[
4x = 68°
\]
\[
x = 17°
\]

Alternate Exterior Angles Theorem

**Example 5:** Find \( m\angle 1 \) and \( m\angle 3 \).
3.2. Properties of Parallel Lines

Solution: \( m\angle 1 = 47^\circ \) because they are vertical angles. Because the lines are parallel, \( m\angle 3 = 47^\circ \) by the Corresponding Angles Theorem. Therefore, \( m\angle 2 = 47^\circ \).

\( \angle 1 \) and \( \angle 3 \) are alternate exterior angles.

Alternate Exterior Angles Theorem: If two parallel lines are cut by a transversal, then the alternate exterior angles are congruent.

The proof of this theorem is very similar to that of the Alternate Interior Angles Theorem and you will be asked to do in the exercises at the end of this section.

Example 6: Algebra Connection Find the measure of each angle and the value of \( y \).

Solution: The given angles are alternate exterior angles. Because the lines are parallel, we can set the expressions equal to each other to solve the problem.

\[
(3y + 53)^\circ = (7y - 55)^\circ \\
108^\circ = 4y \\
27^\circ = y
\]

If \( y = 27^\circ \), then each angle is \( 3(27^\circ) + 53^\circ \), or \( 134^\circ \).

Same Side Interior Angles Theorem

Same side interior angles have a different relationship that the previously discussed angle pairs.

Example 7: Find \( m\angle 2 \).
Solution: Here, $\angle 1 = 66^\circ$ because they are alternate interior angles. $\angle 1$ and $\angle 2$ are a linear pair, so they are supplementary.

\begin{align*}
\angle 1 + \angle 2 &= 180^\circ \\
66^\circ + \angle 2 &= 180^\circ \\
\angle 2 &= 114^\circ
\end{align*}

This example shows that if two parallel lines are cut by a transversal, the same side interior angles are supplementary.

**Same Side Interior Angles Theorem:** If two parallel lines are cut by a transversal, then the same side interior angles are supplementary.

If $l \parallel m$ and both are cut by $t$, then $\angle 3 + \angle 5 = 180^\circ$ and $\angle 4 + \angle 6 = 180^\circ$.

You will be asked to do the proof of this theorem in the review questions.

**Example 8: Algebra Connection** Find the measure of $x$. 

Solution: The given angles are same side interior angles. The lines are parallel, therefore the measures of the angles add up to 180°. Write an equation.

\[(2x + 43)^\circ + (2x - 3)^\circ = 180^\circ\]
\[(4x + 40)^\circ = 180^\circ\]
\[4x = 140^\circ\]
\[x = 35^\circ\]

While you might notice other angle relationships, there are no more theorems to worry about. However, we will continue to explore these other angle relationships. For example, same side exterior angles are also supplementary. You will prove this in the review questions.

Example 9: \(l \parallel m\) and \(s \parallel t\). Prove \(\angle 1 \cong \angle 16\).

Solution:

\[
\begin{array}{|c|c|}
\hline
\text{Statement} & \text{Reason} \\
\hline
1. \(l \parallel m\) and \(s \parallel t\) & Given \\
2. \(\angle 1 \cong \angle 3\) & Corresponding Angles Postulate \\
3. \(\angle 3 \cong \angle 16\) & Alternate Exterior Angles Theorem \\
4. \(\angle 1 \cong \angle 16\) & Transitive PoC \\
\hline
\end{array}
\]

Know What? Revisited Using what we have learned in this lesson, the other angles that are 35° are \(\angle TLQ, \angle ETL,\) and the vertical angle with \(\angle TLQ\). The other angles that are 160° are \(\angle FSR, \angle TSQ,\) and the vertical angle with \(\angle SQV\). You could argue that the State Streets exist to help traffic move faster and more efficiently through the city.

Review Questions

For questions 1-7, determine if each angle pair below is congruent, supplementary or neither.
1. $\angle 1$ and $\angle 7$
2. $\angle 4$ and $\angle 2$
3. $\angle 6$ and $\angle 3$
4. $\angle 5$ and $\angle 8$
5. $\angle 1$ and $\angle 6$
6. $\angle 4$ and $\angle 6$
7. $\angle 2$ and $\angle 3$

For questions 8-16, determine if the angle pairs below are: Corresponding Angles, Alternate Interior Angles, Alternate Exterior Angles, Same Side Interior Angles, Vertical Angles, Linear Pair or None.

8. $\angle 2$ and $\angle 13$
9. $\angle 7$ and $\angle 12$
10. $\angle 1$ and $\angle 11$
11. $\angle 6$ and $\angle 10$
12. $\angle 14$ and $\angle 9$
13. $\angle 3$ and $\angle 11$
14. $\angle 4$ and $\angle 15$
15. $\angle 5$ and $\angle 16$
16. List all angles congruent to $\angle 8$

For 17-20, find the values of $x$ and $y$.

17. $\angle y$ and $\angle 70^\circ$
Algebra Connection For questions 21-25, use the picture shown. Find the value of x and/or y.

21. \( m\angle 1 = (4x + 35)^\circ, m\angle 8 = (7x - 40)^\circ \)
22. \( m\angle 2 = (3y + 14)^\circ, m\angle 6 = (8y - 76)^\circ \)
23. \( m\angle 3 = (3x + 12)^\circ, m\angle 5 = (5x + 8)^\circ \)
24. \( m\angle 4 = (5x - 33)^\circ, m\angle 5 = (2x + 60)^\circ \)
25. \( m\angle 1 = (11y - 15)^\circ, m\angle 7 = (5y + 3)^\circ \)
26. Fill in the blanks in the proof below.

Given: \( l \parallel m \)
Prove: \( \angle 3 \) and \( \angle 5 \) are supplementary (Same Side Interior Angles Theorem). NOTE: You may not use the Same Side Interior Angles Theorem in the proof...remember that theorems must be proved \textit{before} they can be used!

\[ \text{TABLE 3.3:} \]

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td></td>
</tr>
<tr>
<td>2. ( \angle 1 \cong \angle 5 )</td>
<td>Given</td>
</tr>
<tr>
<td>3.</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>( \cong ) angles have = measures</td>
</tr>
<tr>
<td>5.</td>
<td>Linear Pair Postulate</td>
</tr>
<tr>
<td>6. ( m\angle 3 + m\angle 5 = 180^\circ )</td>
<td>Definition of Supplementary Angles</td>
</tr>
<tr>
<td>7. ( \angle 3 ) and ( \angle 5 ) are supplementary</td>
<td></td>
</tr>
</tbody>
</table>

For 27 and 28, use the picture to the right to complete each proof.

27. Given: \( l \parallel m \)
    Prove: \( \angle 1 \cong \angle 8 \) (Alternate Exterior Angles Theorem)

28. Given: \( l \parallel m \)
    Prove: \( \angle 2 \) and \( \angle 8 \) are supplementary

For 29-31, use the picture to the right to complete each proof.

29. Given: \( l \parallel m, s \parallel t \)
    Prove: \( \angle 4 \cong \angle 10 \)
3.2. Properties of Parallel Lines

30. Given: \(l \parallel m, s \parallel t\)
Prove: \(\angle 2 \cong \angle 15\)

31. Given: \(l \parallel m, s \parallel t\)
Prove: \(\angle 4\) and \(\angle 9\) are supplementary

32. Find the measures of all the numbered angles in the figure below.

Algebra Connection
For 33 and 34, find the values of \(x\) and \(y\).

33. (7x -15)°
(3y +12)°

34. (3x + 9)°
(4y + 8)°

35. Error Analysis Nadia is working on Problem 31. Here is her proof:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (l \parallel m, s \parallel t)</td>
<td>Given</td>
</tr>
<tr>
<td>2. (\angle 4 \cong \angle 15)</td>
<td>Alternate Exterior Angles Theorem</td>
</tr>
<tr>
<td>3. (\angle 15 \cong \angle 14)</td>
<td>Same Side Interior Angles Theorem</td>
</tr>
<tr>
<td>4. (\angle 14 \cong \angle 9)</td>
<td>Vertical Angles Theorem</td>
</tr>
<tr>
<td>5. (\angle 4 \cong \angle 9)</td>
<td>Transitive PoC</td>
</tr>
</tbody>
</table>

What happened? Explain what is needed to be done to make the proof correct.

Review Queue Answers

1. \(\angle 1\) and \(\angle 6\), \(\angle 2\) and \(\angle 8\), \(\angle 3\) and \(\angle 7\), or \(\angle 4\) and \(\angle 5\)
2. \(\angle 2\) and \(\angle 5\) or \(\angle 3\) and \(\angle 6\)
3. \(\angle 1\) and \(\angle 7\) or \(\angle 4\) and \(\angle 8\)
4. \( \angle 3 \) and \( \angle 5 \) or \( \angle 2 \) and \( \angle 6 \)
Learning Objectives

- Use the *converses* of the Corresponding Angles Postulate, Alternate Interior Angles Theorem, Alternate Exterior Angles Theorem, and the Same Side Interior Angles Theorem to show that lines are parallel.
- Construct parallel lines using the above converses.
- Use the Parallel Lines Property.

Review Queue

Answer the following questions.

1. Write the converse of the following statements:
   (a) *If it is summer, then I am out of school.*
   (b) *I will go to the mall when I am done with my homework.*
   (c) *If two parallel lines are cut by a transversal, then the corresponding angles are congruent.*

2. Are any of the three converses from #1 true? Why or why not? Give a counterexample.
3. Determine the value of \( x \) if \( l \parallel m \).

Know What? Here is a picture of the support beams for the Coronado Bridge in San Diego. This particular bridge, called a girder bridge, is usually used in straight, horizontal situations. The Coronado Bridge is diagonal, so the beams are subject to twisting forces (called torque). This can be fixed by building a curved bridge deck. To aid the curved bridge deck, the support beams should not be parallel. If they are, the bridge would be too fragile and susceptible to damage.
This bridge was designed so that $\angle 1 = 92^\circ$ and $\angle 2 = 88^\circ$. Are the support beams parallel?

**Corresponding Angles Converse**

Recall that the converse of a statement switches the conclusion and the hypothesis. So, if $a$, then $b$ becomes if $b$, then $a$. We will find the converse of all the theorems from the last section and will determine if they are true.

The Corresponding Angles Postulate says: *If two lines are parallel, then the corresponding angles are congruent.*

The converse is:

**Converse of Corresponding Angles Postulate:** If corresponding angles are congruent when two lines are cut by a transversal, then the lines are parallel.

Is this true? For example, if the corresponding angles both measured $60^\circ$, would the lines be parallel? YES. All eight angles created by $l$, $m$ and the transversal are either $60^\circ$ or $120^\circ$, making the slopes of $l$ and $m$ the same which makes them parallel. This can also be seen by using a construction.

**Investigation 3-5: Creating Parallel Lines using Corresponding Angles**

1. Draw two intersecting lines. Make sure they are not perpendicular. Label them $l$ and $m$, and the point of intersection, $A$, as shown.

![Diagram of two intersecting lines with a point labeled A on line m above line l.]

2. Create a point, $B$, on line $m$, above $A$.

![Diagram showing point B on line m above point A on line l.]
3.3 Proving Lines Parallel

3. Copy the acute angle at $A$ (the angle to the right of $m$) at point $B$. See Investigation 2-2 in Chapter 2 for the directions on how to copy an angle.

![Diagram]

4. Draw the line from the arc intersections to point $B$.

From this construction, we can see that the lines are parallel.

**Example 1:** If $m\angle 8 = 110^\circ$ and $m\angle 4 = 110^\circ$, then what do we know about lines $l$ and $m$?

**Solution:** $\angle 8$ and $\angle 4$ are corresponding angles. Since $m\angle 8 = m\angle 4$, we can conclude that $l \parallel m$.

**Alternate Interior Angles Converse**

We also know, from the last lesson, that when parallel lines are cut by a transversal, the alternate interior angles are congruent. The converse of this theorem is also true:

**Converse of Alternate Interior Angles Theorem:** If two lines are cut by a transversal and alternate interior angles are congruent, then the lines are parallel.

**Example 3:** Prove the Converse of the Alternate Interior Angles Theorem.
Given: \( l \) and \( m \) and transversal \( t \)
\( \angle 3 \cong \angle 6 \)
Prove: \( l \parallel m \)

Solution:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( l ) and ( m ) and transversal ( t ); ( \angle 3 \cong \angle 6 )</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( \angle 3 \cong \angle 2 )</td>
<td>Vertical Angles Theorem</td>
</tr>
<tr>
<td>3. ( \angle 2 \cong \angle 6 )</td>
<td>Transitive PoC</td>
</tr>
<tr>
<td>4. ( l \parallel m )</td>
<td>Converse of the Corresponding Angles Postulate</td>
</tr>
</tbody>
</table>

**Table 3.5:**

**Prove Move: Shorten the names of these theorems.** Discuss with your teacher an appropriate abbreviations. For example, the Converse of the Corresponding Angles Postulate could be Corr \( \angle s \) Converse.

Notice that the Corresponding Angles Postulate was not used in this proof. The Transitive Property is the reason for Step 3 because we do not know if \( l \) is parallel to \( m \) until we are done with the proof. You could conclude that if we are trying to prove two lines are parallel, the converse theorems will be used. And, if we are proving two angles are congruent, we must be given that the two lines are parallel.

**Example 4:** Is \( l \parallel m \)?

**Solution:** First, find \( m \angle 1 \). We know its linear pair is 109°. By the Linear Pair Postulate, these two angles add up to 180°, so \( m \angle 1 = 180° - 109° = 71° \). This means that \( l \parallel m \), by the Converse of the Corresponding Angles Postulate.

**Example 5:** **Algebra Connection** What does \( x \) have to be to make \( a \parallel b \)?

**Solution:** Because these are alternate interior angles, they must be equal for \( a \parallel b \). Set the expressions equal to each other and solve.
3.3. Proving Lines Parallel

\[ 3x + 16° = 5x - 54° \]
\[ 70° = 2x \]
\[ 35° = x \quad \text{To make} \ a \ || \ b, \ x = 35°. \]

Converses of Alternate Exterior Angles & Consecutive Interior Angles Theorems

You have probably guessed that the converse of the Alternate Exterior Angles Theorem and the Consecutive Interior Angles Theorem are also true.

**Converse of the Alternate Exterior Angles Theorem:** If two lines are cut by a transversal and the alternate exterior angles are congruent, then the lines are parallel.

**Example 6: Real-World Situation** The map below shows three roads in Julio's town.

Julio used a surveying tool to measure two angles at the intersections in this picture he drew (NOT to scale). **Julio wants to know if Franklin Way is parallel to Chavez Avenue.**

Solution: The labeled 130° angle and \( \angle a \) are alternate exterior angles. If \( m\angle a = 130° \), then the lines are parallel. To find \( m\angle a \), use the other labeled angle which is 40°, and its linear pair. Therefore, \( \angle a + 40° = 180° \) and \( \angle a = 140° \). 140° \( \neq \) 130°, so Franklin Way and Chavez Avenue are not parallel streets.

The final converse theorem is of the Same Side Interior Angles Theorem. Remember that these angles are not congruent when lines are parallel, they are **supplementary.**

**Converse of the Same Side Interior Angles Theorem:** If two lines are cut by a transversal such that the consecutive interior angles are supplementary, then the lines are parallel.

**Example 7:** Is \( l \ || \ m? \) How do you know?

Solution: These are Same Side Interior Angles. So, if they add up to 180°, then \( l \ || \ m. \ 113° + 67° = 180° \), therefore \( l \ || \ m. \)
Parallel Lines Property

The Parallel Lines Property is a transitive property that can be applied to parallel lines. Remember the Transitive Property of Equality is: If \( a = b \) and \( b = c \), then \( a = c \). The Parallel Lines Property changes \( = \) to \(||\).

**Parallel Lines Property:** If lines \( l || m \) and \( m || n \), then \( l || n \).

**Example 8:** Are lines \( q \) and \( r \) parallel?

**Solution:** First find if \( p || q \), followed by \( p || r \). If so, then \( q || r \).

\( p || q \) by the Converse of the Corresponding Angles Postulate, the corresponding angles are 65°. \( p || r \) by the Converse of the Alternate Exterior Angles Theorem, the alternate exterior angles are 115°. Therefore, by the Parallel Lines Property, \( q || r \).

**Know What? Revisited:** The Coronado Bridge has \( \angle 1 \) and \( \angle 2 \), which are corresponding angles. These angles must be equal for the beams to be parallel. \( \angle 1 = 92° \) and \( \angle 2 = 88° \) and \( 92° \neq 88° \), so the beams are not parallel, therefore a sturdy and safe girder bridge.

**Review Questions**

1. **Construction** Using Investigation 3-1 to help you, show that two lines are parallel by constructing congruent alternate interior angles. HINT: Steps 1 and 2 will be exactly the same, but at step 3, you will copy the angle in a different location.

2. **Construction** Using Investigation 3-1 to help you, show that two lines are parallel by constructing supplementary consecutive interior angles. HINT: Steps 1 and 2 will be exactly the same, but at step 3, you will copy a different angle.

For Questions 3-5, fill in the blanks in the proofs below.

3. Given: \( l || m, p || q \)
   
   Prove: \( \angle 1 \cong \angle 2 \)
3.3. Proving Lines Parallel

TABLE 3.6:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $l \parallel m$</td>
<td>1.</td>
</tr>
<tr>
<td>2.</td>
<td>2. Corresponding Angles Postulate</td>
</tr>
<tr>
<td>3. $p \parallel q$</td>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
<td>4.</td>
</tr>
<tr>
<td>5. $\angle 1 \cong \angle 2$</td>
<td>5.</td>
</tr>
</tbody>
</table>

4. Given: $p \parallel q, \angle 1 \cong \angle 2$
Prove: $l \parallel m$

TABLE 3.7:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $p \parallel q$</td>
<td>1.</td>
</tr>
<tr>
<td>2.</td>
<td>2. Corresponding Angles Postulate</td>
</tr>
<tr>
<td>3. $\angle 1 \cong \angle 2$</td>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
<td>4. Transitive PoC</td>
</tr>
<tr>
<td>5.</td>
<td>5. Converse of Alternate Interior Angles Theorem</td>
</tr>
</tbody>
</table>

5. Given: $\angle 1 \cong \angle 2, \angle 3 \cong \angle 4$
Prove: $l \parallel m$
Table 3.8:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle 1 \cong \angle 2 )</td>
<td>1.</td>
</tr>
<tr>
<td>2. ( l \parallel n )</td>
<td>2.</td>
</tr>
<tr>
<td>3. ( \angle 3 \cong \angle 4 )</td>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
<td>4. Converse of Alternate Interior Angles Theorem</td>
</tr>
<tr>
<td>5. ( l \parallel m )</td>
<td>5.</td>
</tr>
</tbody>
</table>

For Questions 6-9, create your own two column proof.

6. Given: \( m \perp l, n \perp l \)
   Prove: \( m \parallel n \)

7. Given: \( \angle 1 \cong \angle 3 \)
   Prove: \( \angle 1 \) and \( \angle 4 \) are supplementary

8. Given: \( \angle 2 \cong \angle 4 \)
   Prove: \( \angle 1 \cong \angle 3 \)
9. Given: \( \angle 2 \cong \angle 3 \)
    Prove: \( \angle 1 \cong \angle 4 \)

In 10-15, use the given information to determine which lines are parallel. If there are none, write *none*. Consider each question individually.

10. \( \angle LCD \cong \angle CJI \)
11. \( \angle BCE \) and \( \angle BAF \) are supplementary
12. \( \angle FGH \cong \angle EIJ \)
13. \( \angle BFH \cong \angle CEI \)
14. \( \angle LBA \cong \angle IHK \)
15. \( \angle ABG \cong \angle BGH \)

In 16-22, find the measure of the numbered angles below.
16. \( m\angle 1 \)
17. \( m\angle 2 \)
18. \( m\angle 3 \)
19. \( m\angle 4 \)
20. \( m\angle 5 \)
21. \( m\angle 6 \)
22. \( m\angle 7 \)

For 23-27, what does \( x \) have to measure to make the lines parallel?

23. \( m\angle 3 = (3x + 25)\degree \) and \( m\angle 5 = (4x - 55)\degree \)
24. \( m\angle 2 = (8x)\degree \) and \( m\angle 7 = (11x - 36)\degree \)
25. \( m\angle 1 = (6x - 5)\degree \) and \( m\angle 5 = (5x + 7)\degree \)
26. \( m\angle 4 = (3x - 7)\degree \) and \( m\angle 7 = (5x - 21)\degree \)
27. \( m\angle 1 = (9x)\degree \) and \( m\angle 6 = (37x)\degree \)

28. **Construction** Draw a line. Construct a line perpendicular to this line through a point on the line. Now, construct a perpendicular line to this new line. What can you conclude about the original line and this final line?

29. How could you prove your conjecture from problem 28?
30. What is wrong in the following diagram, given that \( j \parallel k \)?
Review Queue Answers

1. (a) *If I am out of school, then it is summer.*
   (b) *If I go to the mall, then I am done with my homework.*
   (c) *If corresponding angles created by two lines cut by a transversal are congruent, then the two lines are parallel.*

2. (a) Not true, I could be out of school on any school holiday or weekend during the school year.
   (b) Not true, I don’t have to be done with my homework to go to the mall.
   (c) Yes, because if two corresponding angles are congruent, then the slopes of these two lines have to be the same, making the lines parallel.

3. The two angles are supplementary.

\[
(17x + 14)° + (4x - 2)° = 180°
\]
\[
21x + 12° = 180°
\]
\[
21x = 168°
\]
\[
x = 8°
\]
3.4 Properties of Perpendicular Lines

Learning Objectives

• Understand the properties of perpendicular lines.
• Explore problems with parallel lines and a perpendicular transversal.
• Solve problems involving complementary adjacent angles.

Review Queue

Determine if the following statements are true or false. If they are true, write the converse. If they are false, find a counterexample.
1. Perpendicular lines form four right angles.
2. A right angle is greater than or equal to 90°.

Find the slope between the two given points.
3. (-3, 4) and (-3, 1)
4. (6, 7) and (-5, 7)

Know What? There are several examples of slope in nature. To the right are pictures of Half Dome, in Yosemite National Park and the horizon over the Pacific Ocean. These are examples of horizontal and vertical lines in real life. Can you determine the slope of these lines?

Congruent Linear Pairs

Recall that a linear pair is a pair of adjacent angles whose outer sides form a straight line. The Linear Pair Postulate says that the angles in a linear pair are supplementary. What happens when the angles in a linear pair are congruent?
3.4. Properties of Perpendicular Lines

**Table 3.9:**

<table>
<thead>
<tr>
<th>(\angle ABD) and (\angle DBC) are supplementary</th>
<th>Linear Pair Postulate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m\angle ABD + m\angle DBC = 180^\circ)</td>
<td>Definition of Supplementary Angles</td>
</tr>
<tr>
<td>(m\angle ABD = m\angle DBC)</td>
<td>Definition of Congruent</td>
</tr>
<tr>
<td>(m\angle ABD + m\angle ABD = 180^\circ)</td>
<td>Substitution PoE</td>
</tr>
<tr>
<td>(2m\angle ABD = 180^\circ)</td>
<td>Combine like terms</td>
</tr>
<tr>
<td>(m\angle ABD = 90^\circ)</td>
<td>Division PoE</td>
</tr>
</tbody>
</table>

So, anytime a linear pair is congruent, the angles will both measure 90°.

**Congruent Linear Pairs Theorem:** If the two angles of a linear pair are congruent, then the measure of each angle is 90°.

**Example 1:** Find \(m\angle CTA\).

**Solution:** First, these two angles form a linear pair. Second, from the marking, we know that \(\angle STC\) is a right angle. Therefore, \(m\angle STC = 90^\circ\). So, \(m\angle CTA\) is also 90°.

**Perpendicular Transversals**

Recall that when two lines intersect, four angles are created. If the two lines are perpendicular, then all four angles are right angles, even though only one needs to be marked with the square. Therefore, all four angles measure 90°.

When a parallel line is added, then there are eight angles formed. If \(l \parallel m\) and \(n \perp l\), is \(n \perp m\)? Let’s prove it here.
Given: \( \overline{l} \parallel \overline{m}, \overline{l} \perp \overline{n} \)

Prove: \( \overline{n} \perp \overline{m} \)

**Table 3.10:**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{l} \parallel \overline{m}, \overline{l} \perp \overline{n} )</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( \angle 1, \angle 2, \angle 3, \text{ and } \angle 4 ) are right angles</td>
<td>Definition of perpendicular lines</td>
</tr>
<tr>
<td>3. ( \angle 1 \equiv \angle 5 )</td>
<td>Definition of a right angle</td>
</tr>
<tr>
<td>4. ( m\angle 1 = m\angle 5 )</td>
<td>Corresponding Angles Postulate</td>
</tr>
<tr>
<td>5. ( \overline{m} \parallel \overline{n} )</td>
<td>Definition of Congruent</td>
</tr>
<tr>
<td>6. ( m\angle 5 = 90^\circ )</td>
<td>Transitive PoE</td>
</tr>
<tr>
<td>7. ( m\angle 6 = m\angle 7 = 90^\circ )</td>
<td>Congruent Linear Pairs</td>
</tr>
<tr>
<td>8. ( \angle 5 \equiv \angle 8 )</td>
<td>Vertical Angles Theorem</td>
</tr>
<tr>
<td>9. ( m\angle 5 = m\angle 8 )</td>
<td>Definition of Congruent</td>
</tr>
<tr>
<td>10. ( m\angle 8 = 90^\circ )</td>
<td>Transitive PoE</td>
</tr>
<tr>
<td>11. ( \angle 5, \angle 6, \angle 7, \text{ and } \angle 8 ) are right angles</td>
<td>Definition of right angle</td>
</tr>
<tr>
<td>12. ( \overline{n} \perp \overline{m} )</td>
<td>Definition of perpendicular lines</td>
</tr>
</tbody>
</table>

**Theorem 3-1:** If two lines are parallel and a third line is perpendicular to one of the parallel lines, it is also perpendicular to the other parallel line.

Or, if \( \overline{l} \parallel \overline{m} \) and \( \overline{l} \perp \overline{n} \), then \( \overline{n} \perp \overline{m} \).

**Theorem 3-2:** If two lines are perpendicular to the same line, they are parallel to each other.

Or, if \( \overline{l} \perp \overline{n} \) and \( \overline{n} \perp \overline{m} \), then \( \overline{l} \parallel \overline{m} \). You will prove this theorem in the review questions.

From these two theorems, we can now assume that any angle formed by two parallel lines and a perpendicular transversal will always measure \( 90^\circ \).

**Example 2:** Determine the measure of \( \angle 1 \).
3.4. Properties of Perpendicular Lines

Solution: From Theorem 3-1, we know that the lower parallel line is also perpendicular to the transversal. Therefore, $m\angle 1 = 90^\circ$.

Adjacent Complementary Angles

Recall that complementary angles are angle pairs whose measures add up to $90^\circ$. If complementary angles are adjacent, their nonadjacent sides are perpendicular rays. What you have learned about perpendicular lines can be applied to this situation.

Example 3: Find $m\angle 1$.

Solution: The two adjacent angles add up to $90^\circ$, so $l \perp m$. Therefore, $m\angle 1 = 90^\circ$.

Example 4: Is $l \perp m$? Explain why or why not.

Solution: If the two adjacent angles add up to $90^\circ$, then $l$ and $m$ are perpendicular.

$23^\circ + 67^\circ = 90^\circ$. Therefore, $l \perp m$.

Know What? Revisited
Half Dome is vertical and the slope of any vertical line is undefined. Thousands of people flock to Half Dome to attempt to scale the rock. This front side is very difficult to climb because it is vertical. The only way to scale the front side is to use the provided cables at the base of the rock. http://www.nps.gov/yose/index.htm

Any horizon over an ocean is horizontal, which has a slope of zero, or no slope. There is no steepness, so no incline or decline. The complete opposite of Half Dome. Actually, if Half Dome was placed on top of an ocean or flat ground, the two would be perpendicular!

**Review Questions**

Find the measure of $\angle 1$ for each problem below.
For questions 10-13, use the picture below.
10. Find $\angle ACD$.
11. Find $\angle CDB$.
12. Find $\angle EDB$.
13. Find $\angle CDE$.

In questions 14-17, determine if $l \perp m$.

For questions 18-25, use the picture below.
18. Find $m\angle 1$.
19. Find $m\angle 2$.
20. Find $m\angle 3$.
21. Find $m\angle 4$.
22. Find $m\angle 5$.
23. Find $m\angle 6$.
24. Find $m\angle 7$.
25. Find $m\angle 8$.

Complete the proof.

26. Given: \( l \perp m, l \perp n \)
Prove: \( m \parallel n \)

Algebra Connection Find the value of \( x \).
Review Queue Answers

1. True; If four right angles are formed by two intersecting lines, then the lines are perpendicular.
2. False; 95° is not a right angle.
3. Undefined slope; this is a vertical line.
4. Zero slope; this would be a horizontal line.
3.5 Parallel and Perpendicular Lines in the Coordinate Plane

Learning Objectives

- Compute slope.
- Determine the equations of lines parallel and perpendicular to a given line.
- Graph parallel and perpendicular lines given in slope-intercept and standard form.

Review Queue

Find the slope between the following points.

1. (-3, 5) and (2, -5)
2. (7, -1) and (-2, 2)
3. Is \( x = 3 \) horizontal or vertical? How do you know?

Graph the following lines on an \( xy \) plane.

4. \( y = -2x + 3 \)
5. \( y = \frac{1}{3}x - 2 \)

Know What? The picture to the right is the California Incline, a short piece of road that connects Highway 1 with the city of Santa Monica. The length of the road is 1532 feet and has an elevation of 177 feet. You may assume that the base of this incline is sea level, or zero feet. Can you find the slope of the California Incline?

HINT: You will need to use the Pythagorean Theorem, which has not been introduced in this class, but you may have seen it in a previous math class.

Slope in the Coordinate Plane

Recall from Algebra I, The slope of the line between two points \( (x_1, y_1) \) and \( (x_2, y_2) \) is \( m = \frac{y_2 - y_1}{x_2 - x_1} \).

Different Types of Slope:
Example 1: What is the slope of the line through (2, 2) and (4, 6)?

Solution: Use the slope formula to determine the slope. Use (2, 2) as \((x_1, y_1)\) and (4, 6) as \((x_2, y_2)\).

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 2}{4 - 2} = \frac{4}{2} = 2 \]

Therefore, the slope of this line is 2.

This slope is positive. Recall that slope can also be thought of as "rise" over "run." In this case we rise, or go up 2, and run in the positive direction 1.

Example 2: Find the slope between (-8, 3) and (2, -2).

Solution: \[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 3}{2 - (-8)} = \frac{-5}{10} = -\frac{1}{2} \]

This is a negative slope. Instead of rising, the negative slope means that you would fall, when finding points on the line.

Example 3: Find the slope between (-5, -1) and (3, -1).
3.5. Parallel and Perpendicular Lines in the Coordinate Plane

Solution:

\[ m = \frac{-1 - (-1)}{3 - (-5)} = \frac{0}{8} = 0 \]

Therefore, the slope of this line is 0, which means that it is a horizontal line. Horizontal lines always pass through the \( y \)-axis. Notice that the \( y \)-coordinate for both points is -1. In fact, the \( y \)-coordinate for any point on this line is -1. This means that the horizontal line must cross \( y = -1 \).

**Example 4:** What is the slope of the line through (3, 2) and (3, 6)?

Solution:

\[ m = \frac{6 - 2}{3 - 3} = \frac{4}{0} = undefined \]

Therefore, the slope of this line is undefined, which means that it is a *vertical* line. Vertical lines always pass through the \( x \)-axis. Notice that the \( x \)-coordinate for both points is 3. In fact, the \( x \)-coordinate for any point on this line is 3. This means that the vertical line must cross \( x = 3 \).
Slopes of Parallel Lines

Recall from earlier in the chapter that the definition of parallel is two coplanar lines that never intersect. In the coordinate plane, that would look like this:

If we take a closer look at these two lines, we see that the slopes of both are $\frac{2}{3}$.
This can be generalized to any pair of parallel lines in the coordinate plane.

*Parallel lines have the same slope.*

**Example 5:** Find the equation of the line that is parallel to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

Recall that the equation of a line in this form is called the slope-intercept form and is written as $y = mx + b$ where $m$ is the slope and $b$ is the $y$—intercept. Here, $x$ and $y$ represent any coordinate pair, $(x, y)$ on the line.

**Solution:** We know that parallel lines have the same slope, so the line we are trying to find also has $m = -\frac{1}{3}$. Now, we need to find the $y$—intercept. 4 is the $y$—intercept of the given line, *not our new line*. We need to plug in 9 for $x$ and -5 for $y$ (this is our given coordinate pair that needs to be on the line) to solve for the *new* $y$—intercept ($b$).

\[-5 = -\frac{1}{3}(9) + b\]
\[-5 = -3 + b\]
Therefore, the equation of line is $y = -\frac{1}{3}x - 2$.

\[-2 = b\]

Reminder: the final equation contains the variables $x$ and $y$ to indicate that the line contains and infinite number of points or coordinate pairs that satisfy the equation.

*Parallel lines always have the same slope and different y-intercepts.*

Slopes of Perpendicular Lines

Recall from an earlier chapter that the definition of perpendicular is two lines that intersect at a $90^\circ$, or right, angle. In the coordinate plane, that would look like this:
3.5. Parallel and Perpendicular Lines in the Coordinate Plane

If we take a closer look at these two lines, we see that the slope of one is -4 and the other is $\frac{1}{4}$.

This can be generalized to any pair of perpendicular lines in the coordinate plane.

The slopes of perpendicular lines have opposite signs and are reciprocals of each other. The product of the slopes of perpendicular lines is therefore -1.

**Example 6:** Find the slope of the perpendicular lines to the lines below.

a) $y = 2x + 3$

b) $y = -\frac{2}{3}x - 5$

c) $y = x + 2$

**Solution:** We are only concerned with the slope for each of these.

a) $m = 2$, so $m_\perp$ is the reciprocal and negative, $m_\perp = -\frac{1}{2}$.

b) $m = -\frac{2}{3}$, take the reciprocal and make the slope positive, $m_\perp = \frac{3}{2}$.

c) Because there is no number in front of $x$, the slope is 1. The reciprocal of 1 is 1, so the only thing to do is make it negative, $m_\perp = -1$.

**Example 7:** Find the equation of the line that is perpendicular to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

**Solution:** First, the slope is the reciprocal and opposite sign of $-\frac{1}{3}$. So, $m = 3$. Now, we need to find the $y-$intercept. 4 is the $y-$intercept of the given line, not our new line. We need to plug in 9 for $x$ and -5 for $y$ to solve for the new $y-$intercept ($b$).

\[-5 = 3(9) + b \]
\[-5 = 27 + b \quad \text{Therefore, the equation of line is } y = 3x - 32.\]
\[-32 = b \]

**Graphing Parallel and Perpendicular Lines**

**Example 8:** Find the equations of the lines below and determine if they are parallel, perpendicular or neither.

![Graph of parallel and perpendicular lines]
Solution: To find the equation of each line, start with the $y$–intercept. The top line has a $y$–intercept of 1. From there, determine the slope triangle, or the rise over run. From the $y$–intercept, if you go up 1 and over 2, you hit the line again. Therefore, the slope of this line is $\frac{1}{2}$. The equation is $y = \frac{1}{2}x + 1$. For the second line, the $y$–intercept is -3. Again, start here to determine the slope and if you rise 1 and run 2, you run into the line again, making the slope $\frac{1}{2}$. The equation of this line is $y = \frac{1}{2}x - 3$. The lines are parallel because they have the same slope.

Example 9: Graph $3x - 4y = 8$ and $4x + 3y = 15$. Determine if they are parallel, perpendicular, or neither.

Solution: First, we have to change each equation into slope-intercept form. In other words, we need to solve each equation for $y$.

$$
3x - 4y = 8 \\
-4y = -3x + 8 \\
y = \frac{3}{4}x - 2
$$

$$
4x + 3y = 15 \\
3y = -4x + 15 \\
y = -\frac{4}{3}x + 5
$$

Now that the lines are in slope-intercept form (also called $y$–intercept form), we can tell they are perpendicular because the slopes have opposites signs and are reciprocals.

To graph the two lines, plot the $y$–intercept on the $y$–axis. From there, use the slope to rise and then run. For the first line, you would plot -2 and then rise 3 and run 4, making the next point on the line (1, 4). For the second line, plot 5 and then fall (because the slop is negative) 4 and run 3, making the next point on the line (1, 3).
Know What? Revisited In order to find the slope, we need to first find the horizontal distance in the triangle to the right. This triangle represents the incline and the elevation. To find the horizontal distance, or the run, we need to use the Pythagorean Theorem, \( a^2 + b^2 = c^2 \), where \( c \) is the hypotenuse.

The slope is then \( \frac{177}{1521.741} \), or .116, which is roughly \( \frac{29}{250} \).

Review Questions

Find the slope between the two given points.

1. (4, -1) and (-2, -3)
2. (-9, 5) and (-6, 2)
3. (7, 2) and (-7, -2)
4. (-6, 0) and (-1, -10)
5. (1, -2) and (3, 6)
6. (-4, 5) and (-4, -3)

Determine if each pair of lines are parallel, perpendicular, or neither. Then, graph each pair on the same set of axes.

7. \( y = -2x + 3 \) and \( y = \frac{1}{2}x + 3 \)
8. \( y = 4x - 2 \) and \( y = 4x + 5 \)
9. \( y = -x + 5 \) and \( y = x + 1 \)
10. \( y = -3x + 1 \) and \( y = 3x - 1 \)
11. \( 2x - 3y = 6 \) and \( 3x + 2y = 6 \)
12. \( 5x + 2y = -4 \) and \( 5x + 2y = 8 \)
13. \( x - 3y = -3 \) and \( x + 3y = 9 \)
14. \( x + y = 6 \) and \( 4x + 4y = -16 \)

Determine the equation of the line that is parallel to the given line, through the given point.

15. \( y = -5x + 1; \ (−2, 3) \)
16. \( y = \frac{2}{3}x - 2; \ (9, 1) \)
17. \( x - 4y = 12; \ (-16, -2) \)
18. \( 3x + 2y = 10; \ (8, -11) \)
19. \( 2x - y = 15; \ (3, 7) \)
20. \( y = x - 5; \ (9, -1) \)

Determine the equation of the line that is perpendicular to the given line, through the given point.

21. \( y = x - 1; \ (-6, 2) \)
22. \( y = 3x + 4; \ (9, -7) \)
23. \( 5x - 2y = 6; \ (5, 5) \)
24. \( y = 4; \ (-1, 3) \)
25. \( x = -3; \ (1, 8) \)
26. \( x - 3y = 11; \ (0, 13) \)

Find the equation of the two lines in graph below. Then, determine if the two lines are parallel, perpendicular or neither.
For the line and point below, find:

a) A parallel line, through the given point.

b) A perpendicular line, through the given point.
Review Queue Answers

1. \( m = \frac{-5 - 5}{2 + 3} = \frac{-10}{5} = -2 \)
2. \( m = \frac{2 - 1}{3 - 2} = \frac{1}{1} = 1 \)
3. Vertical because it has to pass through \( x = 3 \) on the \( x \)-axis and doesn’t pass through the \( y \) axis at all.
Learning Objectives

- Find the distance between two points.
- Find the shortest distance between a point and a line and two parallel lines.
- Determine the equation of a perpendicular bisector of a line segment in the coordinate plane.

Review Queue

1. What is the equation of the line between (-1, 3) and (2, -9)?
2. Find the equation of the line that is perpendicular to \( y = -2x + 5 \) through the point (-4, -5).
3. Find the equation of the line that is parallel to \( y = \frac{2}{3}x - 7 \) through the point (3, 8).

Know What? The shortest distance between two points is a straight line. To the right is an example of how far apart cities are in the greater Los Angeles area. There are always several ways to get somewhere in Los Angeles. Here, we have the distances between Los Angeles and Orange. Which distance is the shortest? Which is the longest?

The Distance Formula

The distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) can be defined as \(d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\). This formula will be derived in a later chapter.

Example 1: Find the distance between (4, -2) and (-10, 3).

Solution: Plug in (4, -2) for \((x_1, y_1)\) and (-10, 3) for \((x_2, y_2)\) and simplify.
\[ d = \sqrt{(-10 - 4)^2 + (3 + 2)^2} \]
\[ = \sqrt{(-14)^2 + (5^2)} \quad \text{Distances are always positive!} \]
\[ = \sqrt{196 + 25} \]
\[ = \sqrt{221} \approx 14.866 \text{ units} \]

**Example 2:** The distance between two points is 4 units. One point is (1, -6). What is the second point? You may assume that the second point is made up of integers.

**Solution:** We will still use the distance formula for this problem, however, we know \( d \) and need to solve for \((x_2, y_2)\).

\[ 4 = \sqrt{(1 - x_2)^2 + (-6 - y_2)^2} \]
\[ 16 = (1 - x_2)^2 + (-6 - y_2)^2 \]

At this point, we need to figure out two square numbers that add up to 16. The only two square numbers that add up to 16 are 16 + 0.

\[ 16 = (1 - x_2)^2 + (-6 - y_2)^2 \quad \text{or} \quad 16 = 16 + 0 \]
\[ 1 - x_2 = \pm 4 \quad \text{or} \quad 1 - x_2 = 0 \quad \text{and} \quad -6 - y_2 = 0 \quad \text{or} \quad -6 - y_2 = \pm 4 \]
\[ x_2 = 5 \text{ or } -3 \quad \text{and} \quad y_2 = -6 \quad \text{or} \quad y_2 = 10 \text{ or } -2 \]

Therefore, the second point could have 4 possibilities: (5, -6), (-3, -6), (1, -10), and (1, -2).

**Shortest Distance between a Point and a Line**

We know that the shortest distance between two points is a straight line. This distance can be calculated by using the distance formula. Let’s extend this concept to the shortest distance between a point and a line.

![Diagram of point A and line l with various line segments showing shortest distances.]

Just by looking at a few line segments from A to line \( l \), we can tell that the shortest distance between a point and a line is the perpendicular line between them. Therefore, \( AD \) is the shortest distance between A and line \( l \).

Putting this onto a graph can be a little tougher.

**Example 3:** Determine the shortest distance between the point (1, 5) and the line \( y = \frac{1}{3}x - 2 \).
3.6. The Distance Formula

Solution: First, graph the line and point. Second determine the equation of the perpendicular line. The opposite sign and reciprocal of $\frac{1}{3}$ is -3, so that is the slope. We know the line must go through the given point, (1, 5), so use that to find the y-intercept.

\[ y = -3x + b \]
\[ 5 = -3(1) + b \]
\[ 8 = b \]

The equation of the line is $y = -3x + 8$.

Next, we need to find the point of intersection of these two lines. By graphing them on the same axes, we can see that the point of intersection is (3, -1), the green point.

Finally, plug (1, 5) and (3,-1) into the distance formula to find the shortest distance.
\[ d = \sqrt{(3 - 1)^2 + (-1 - 5)^2} \]
\[ = \sqrt{2^2 + (-6)^2} \]
\[ = \sqrt{2 + 36} \]
\[ = \sqrt{38} \approx 6.164 \text{ units} \]

**Shortest Distance between Two Parallel Lines**

The shortest distance between two parallel lines is the length of the perpendicular segment between them. It doesn’t matter which perpendicular line you choose, as long as the two points are on the lines. Recall that there are infinitely many perpendicular lines between two parallel lines.

Notice that all of the pink segments are the same length. So, when picking a perpendicular segment, be sure to pick one with endpoints that are integers.

**Example 3:** Find the distance between \( x = 3 \) and \( x = -5 \).

**Solution:** Any line with \( x = a \) number is a vertical line. In this case, we can just count the squares between the two lines. The two lines are \( 3 - (-5) \) units apart, or 8 units.

You can use this same method with horizontal lines as well. For example, \( y = -1 \) and \( y = 3 \) are \( 3 - (-1) \) units, or 4 units apart.

**Example 4:** What is the shortest distance between \( y = 2x + 4 \) and \( y = 2x - 1 \)?
Solution: Graph the two lines and determine the perpendicular slope, which is $-\frac{1}{2}$. Find a point on $y = 2x + 4$, lets say (-1, 2). From here, use the slope of the perpendicular line to find the corresponding point on $y = 2x - 1$. If you move down 1 from 2 and over to the right 2 from -1, you will hit $y = 2x - 1$ at (1, 1). Use these two points to determine the distance between the two lines.

\[
d = \sqrt{(1+1)^2 + (1-2)^2} = \sqrt{2^2 + (-1)^2} = \sqrt{4 + 1} = \sqrt{5} \approx 2.236 \text{ units}
\]

The lines are about 2.236 units apart.

Notice that you could have used any two points, as long as they are on the same perpendicular line. For example, you could have also used (-3, -2) and (-1, -3) and you still would have gotten the same answer.
\[ d = \sqrt{(-1 + 3)^2 + (-3 + 2)^2} \]
\[ = \sqrt{2^2 + (-1)^2} \]
\[ = \sqrt{4 + 1} \]
\[ = \sqrt{5} \approx 2.236 \text{ units} \]

**Example 5:** Find the distance between the two parallel lines below.

**Solution:** First you need to find the slope of the two lines. Because they are parallel, they are the same slope, so if you find the slope of one, you have the slope of both.

Start at the \( y \)--intercept of the top line, 7. From there, you would go down 1 and over 3 to reach the line again. Therefore the slope of both parallel lines is \(-\frac{1}{3}\) and the perpendicular line’s slope would be 3.

Next, find two points on the lines. Let's use the \( y \)--intercept of the bottom line, (0, -3). Then, rise 3 and go over 1 until your reach the second line. Doing this three times, you would hit the top line at (3, 6). Use these two points in the distance formula to find how far apart the lines are.
3.6. The Distance Formula

\[ d = \sqrt{(0-3)^2 + (-3-6)^2} \]
\[ = \sqrt{(-3)^2 + (-9)^2} \]
\[ = \sqrt{9 + 81} \]
\[ = \sqrt{90} \approx 9.487 \text{ units} \]

**Perpendicular Bisectors in the Coordinate Plane**

Recall that the definition of a perpendicular bisector is a perpendicular line that goes through the midpoint of a line segment. Using what we have learned in this chapter and the formula for a midpoint, we can find the equation of a perpendicular bisector.

**Example 6:** Find the equation of the perpendicular bisector of the line segment between (-1, 8) and (5, 2).

**Solution:**

First, find the midpoint of the line segment.

\[ \left( \frac{-1 + 5}{2}, \frac{8 + 2}{2} \right) = \left( \frac{4}{2}, \frac{10}{2} \right) = (2, 5) \]

Second, find the slope between the two endpoints. This will help us figure out the perpendicular slope for the perpendicular bisector.

\[ m = \frac{2 - 8}{5 + 1} = \frac{-6}{6} = -1 \]

If the slope of the segment is -1, then the slope of the perpendicular bisector will be 1. The last thing to do is to find the y–intercept of the perpendicular bisector. We know it goes through the midpoint, (2, 5), of the segment, so substitute that in for \( x \) and \( y \) in the slope-intercept equation.
The equation of the perpendicular bisector is \( y = x + 3 \).

**Example 7:** The perpendicular bisector of \( \overline{AB} \) has the equation \( y = -\frac{4}{3}x + 1 \). If \( A \) is \((-1, 8)\) what are the coordinates of \( B \)?

**Solution:** The easiest way to approach this problem is to graph it. Graph the perpendicular line and plot the point. See the graph to the left.

Second, determine the slope of \( \overline{AB} \). If the slope of the perpendicular bisector is \(-\frac{1}{3}\), then the slope of \( \overline{AB} \) is 3.

Using the slope, count down 3 and over to the right 1 until you hit the perpendicular bisector. Counting down 6 and over 2, you land on the line at \((-3, 2)\). This is the midpoint of \( \overline{AB} \). If you count down another 6 and over to the right 2 more, you will find the coordinates of \( B \), which are \((-5, -4)\).
Know What? Revisited Draw two intersecting lines. Make sure they are not perpendicular. Label the 26.3 miles along hwy 5. The longest distance is found by adding the distances along the 110 and 405, or 41.8 miles.

Review Questions

Find the distance between each pair of points. Round your answer to the nearest thousandth.

1. (4, 15) and (-2, -1)
2. (-6, 1) and (9, -11)
3. (0, 12) and (-3, 8)
4. (-8, 19) and (3, 5)
5. (3, -25) and (-10, -7)
6. (-1, 2) and (8, -9)
7. (5, -2) and (1, 3)
8. (-30, 6) and (-23, 0)

Determine the shortest distance between the given line and point. Round your answers to the nearest thousandth.

9. \( y = \frac{1}{3}x + 4; \ (5, -1) \)
10. \( y = 2x - 4; \ (-7, -3) \)
11. \( y = -4x + 1; \ (4, 2) \)
12. \( y = -\frac{3}{2}x - 8; \ (7, 9) \)

Use each graph below to determine how far apart each the parallel lines are. Round your answers to the nearest thousandth.
Determine the shortest distance between the each pair of parallel lines. Round your answer to the nearest thousandth.

17. \(x = 5, \ x = 1\)
18. \(y = -6, \ y = 4\)
19. \(y = x + 5, \ y = x - 3\)
20. \( y = -\frac{1}{3}x + 2, \) \( y = -\frac{1}{3}x - 8 \)
21. \( y = 4x + 9, \) \( y = 4x - 8 \)
22. \( y = \frac{1}{2}x, \) \( y = \frac{1}{2}x - 5 \)

For questions 23-26, find the equation of the perpendicular bisector for each pair of points.

23. (1, 5) and (7, -7)
24. (1, -8) and (7, -6)
25. (9, 2) and (-9, -10)
26. (-7, 11) and (-3, 1)
27. The perpendicular bisector of \( \overline{CD} \) has the equation \( y = 3x - 11 \). If \( D \) is (-3, 0) what are the coordinates of \( C \)?
28. The perpendicular bisector of \( \overline{LM} \) has the equation \( y = -x + 5 \). If \( L \) is (6, -3) what are the coordinates of \( M \)?
29. \textbf{Construction} Plot the points (5, -3) and (-5, -9). Draw the line segment between the points. Construct the perpendicular bisector for these two points. (Construction was in Chapter 1). Determine the equation of the perpendicular bisector and the midpoint.
30. \textbf{Construction} Graph the line \( y = -\frac{1}{2}x - 5 \) and the point (2, 5). Construct the perpendicular line, through (2, 5) and determine the equation of this line.
31. \textbf{Challenge} The distance between two points is 25 units. One point is (-2, 9). What is the second point? You may assume that the second point is made up of integers.
32. \textbf{Writing} List the steps you would take to find the distance between two parallel lines, like the two in #24.

\textbf{Review Queue Answers}

1. \( y = -4x - 1 \)
2. \( y = \frac{1}{2}x - 3 \)
3. \( y = \frac{2}{3}x + 6 \)
Keywords and Theorems

- Parallel
- Skew Lines
- Parallel Postulate
- Perpendicular Line Postulate
- Transversal
- Corresponding Angles
- Alternate Interior Angles
- Alternate Exterior Angles
- Same Side Interior Angles
- Corresponding Angles Postulate
- Alternate Interior Angles Theorem
- Alternate Exterior Angles Theorem
- Same Side Interior Angles Theorem
- Converse of Corresponding Angles Postulate
- Converse of Alternate Interior Angles Theorem
- Converse of the Alternate Exterior Angles Theorem
- Converse of the Same Side Interior Angles Theorem
- Parallel Lines Property
- Congruent Linear Pairs Theorem
- Theorem 3-1
- Theorem 3-2
- Distance Formula: \[d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\]

Review

Find the value of each of the numbered angles below.
Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9688.